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A note on a swimming problem

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The rate of self-propulsion of a doubly-infinite flexible sheet due to transverse waving oscillations in a viscous fluid is shown to decrease with increasing frequency, at a fixed (small) wave amplitude. This result differs from that of Reynolds (1965) who included local inertia, and thereby predicted that the swimming speed increases above the limiting value given by Taylor (1951) at zero frequency. The error in Reynolds' work is due to his neglect of the simultaneous effect of convection, which induces a non-uniform mean second-order flow, with direction such as to oppose propulsion. Some other results concerning swimming sheets are presented.

1. Introduction

In order to investigate the mechanism of swimming of micro-organisms, Taylor (1951) took as his first model a doubly infinite sheet, flexible but inextensible, which is propelling itself by small transverse oscillations. Taylor considered a wave of displacement $y = b\sin(kx-\sigma t)$ propagating in the $+x$ direction with phase velocity $c = \sigma/k$, and found that this motion induces a velocity in the fluid at infinity of

$$U_\infty = c[\tfrac{1}{2}(kb)^2 + O(kb)^4], \tag{1.1}$$

also in the $+x$ direction. A limitation of Taylor's analysis is that the Reynolds number $R = \sigma/\nu k^2$ based on the phase velocity of the wave must be small enough for the application of the Stokes equations for steady flow. This limitation was removed by Reynolds (1965), who derived, in effect, a multiplicative correction factor $F(R)$, such that

$$U_\infty = c[\tfrac{1}{2}(kb)^2 F(R) + O(kb)^4]. \tag{1.2}$$

In fact, the formula given by Reynolds can be simplified considerably, and reduces to

$$F(R) = \left[\frac{1+(1+R^2)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}, \tag{1.3}$$

a function which increases monotonically from unity at $R = 0$, tending to infinity like $R^{\frac{1}{2}}$ as $R \to \infty$. Thus the effect of inertia appears to be to increase the propulsion velocity above that found by Taylor at $R = 0$.

It is the purpose of the present note to observe that Reynolds' formula (1.2) is incorrect, and that the correct result is

$$U_\infty = c\left[\tfrac{1}{2}(kb)^2\frac{1+F(R)}{2F(R)} + O(kb)^4\right], \tag{1.4}$$

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with the same value (1·3) for $F(R)$. The new expression (1·4) for the velocity of propulsion is in sharp contrast with the previous result (1.2), for now the propulsion velocity *decreases* as R increases, tending to one-half of Taylor's value as $R \to \infty$. The discrepancy between (1·2) and (1.4) arises from the fact that in obtaining (1.2) no account was taken of (second-order) convection terms in the Navier–Stokes equation, which tend to hamper propulsion, whereas the first-order inertia terms (involving $\partial/\partial t$ or $c(\partial/\partial x)$) enhance propulsion. Reynolds (1965, p. 244) anticipates that the mean second-order flow is purely uniform, i.e. independent of x *and* y. In fact the convection terms (written out as the Jacobian on the right of his inhomogeneous Oseen equation (8), but not used in his subsequent analysis) induce a contribution to the mean second-order flow which is independent of x but an exponentially decaying function of distance y from the plate. The analysis in Reynolds' section 3, concerning a standing wave motion, is also in error because of neglect of the convection terms.

2. Analysis

Although we could use the same notation and formulation as Reynolds, an abbreviated alternative derivation is presented here as a matter of interest. We use a stream function ψ satisfying $u = \psi_y$, $v = -\psi_x$, $\omega = -\nabla^2\psi$, and the Navier–Stokes equation

$$\nu\nabla^2\omega - \frac{\partial\omega}{\partial t} = u\omega_x + v\omega_y. \tag{2.1}$$

The boundary conditions (Taylor 1951) are

$$\left.\begin{aligned} u &= \tfrac{1}{4}b^2k\sigma\cos(2kx-2\sigma t) + O(b^4), \\ v &= -\sigma b\cos(kx-\sigma t) + O(b^3), \end{aligned}\right\} \tag{2.2}$$

on the moving surface $\quad y = b\sin(kx-\sigma t).$

We now make the expansion

$$\psi = \mathscr{R}[\psi_1(y)\,e^{-ikx+i\sigma t}] + \Psi_2(y) + \mathscr{R}[\psi_2(y)\,e^{-2ikx+2i\sigma t}] + O(b^3), \tag{2.3}$$

where the first term of (2.3) is $O(b)$ and satisfies a linearized version of the Navier–Stokes equation (2.1), while the remaining second-order terms are divided into a 'D.C.' part $\Psi_2(y) = O(b^2)$ independent of t and x, and a second-harmonic part which varies sinusoidally in t and x, and with which we shall not be concerned.

The solution for the linearized flow is obtained by inspection, with the result

$$\psi_1 = -\nu bl(l+k)\left[\frac{e^{-ly}}{l} - \frac{e^{-ky}}{k}\right], \tag{2.4}$$

where $l = \{k^2 + (i\sigma/\nu)\}^{\frac{1}{2}}$ ($= k\beta e^{i\phi}$ in Reynolds' notation). Although the notation differs substantially, equation (2.4) agrees with Reynolds' equation (12).

The equation satisfied by the D.C. second approximation is

$$\nu\frac{d^4\Psi_2}{dy^4} = -\langle u_1\omega_{1x} + v_1\omega_{1y}\rangle, \tag{2.5}$$

where $\langle\ \rangle$ denotes an average with respect to t or x, and u_1, v_1, ω_1 are first-order quantities. The right-hand side of (2.5) may be evaluated using the above formula (2.4) for ψ_1, with the result

$$\frac{d^4\Psi_2}{dy^4} = \tfrac{1}{2}\sigma b^2|\alpha|^2\mathscr{R}[l\alpha\, e^{-\alpha y} - k\gamma\, e^{-\gamma y}], \tag{2.6}$$

where $\alpha = k+\bar{l}$, $\gamma = l+\bar{l} = 2\mathscr{R}(l)$. The solution for Ψ_2 which corresponds to a velocity U_∞ at $y = \infty$ is

$$\Psi_2 = U_\infty y + \tfrac{1}{2}\sigma b^2|\alpha|^2\mathscr{R}\left[\frac{l}{\alpha^3}e^{-\alpha y} - \frac{k}{\gamma^3}e^{-\gamma y}\right]. \tag{2.7}$$

The boundary condition to be satisfied on $y = 0$ is obtained by substitution of the expansion (2.3) into the boundary conditions (2.2), resulting in

$$\frac{d\Psi_2}{dy} = -\langle y u_{1y}\rangle = \tfrac{1}{4}\sigma b^2\gamma, \tag{2.8}$$

which is consistent with the above solution for Ψ_2 only if

$$U_\infty = \tfrac{1}{4}\sigma b^2\gamma + \tfrac{1}{2}\sigma b^2|\alpha|^2\mathscr{R}\left[\frac{l}{\alpha^2} - \frac{k}{\gamma^2}\right]. \tag{2.9}$$

This expression reduces to equation (1.4) after some manipulation, with the identification

$$\begin{aligned} F(R) &= \frac{\gamma}{2k} \\ &= \frac{1}{k}\mathscr{R}(l) \\ &= \mathscr{R}(1+iR)^{\frac{1}{2}}. \end{aligned}$$

If we neglect the second term of (2.9), we are left with equation (1.2), which can, also after some manipulation, be shown to be equivalent to Reynolds' result.

3. Further comments

Although the unrealistic geometry of the oscillating doubly infinite sheet makes its application as a model for swimming organisms questionable, it may be worthwhile to quote some further results which can easily be expressed in the present notation. Other extensions were discussed by Reynolds (1965).

First, we may observe that Taylor's inextensibility condition is not strictly necessary in the present problem. Thus we made no use of the second-harmonic term in the boundary condition (2.2) for u. The present results are in fact valid for any predominantly transverse waving oscillation of a flexible sheet, such that a particle of the sheet lies at position (x, y) at time t, where

$$x = x_0 + o(b), \quad \langle x - x_0\rangle = o(b^2), \tag{3.1}$$

$$y = b\sin(kx_0 - \sigma t) + o(b^2). \tag{3.2}$$

Here x_0 is a label co-ordinate of the particle, in the Lagrangian sense. Notice that the shape of the sheet need not be exactly sinusoidal in space or time, so that we cannot reduce the problem exactly to a steady flow.

By the same token we could consider wave motions which are predominantly longitudinal, involving the creation of an effective velocity of slip over the sheet. Thus if

$$x = x_0 + b\sin(kx_0 - \sigma t) + o(b^2), \tag{3.3}$$

$$y = o(b^2), \tag{3.4}$$

then the resulting velocity of propulsion is found by a very similar analysis to be

$$U_\infty = -c\left[\tfrac{1}{2}(kb)^2\frac{3F(R)-1}{2F(R)} + o(kb)^2\right]. \tag{3.5}$$

Notice that the fluid at infinity is induced to move in a direction *opposite* to that of the wave, at a rate which *increases* with R. Reynolds (1965) considered cases intermediate between transverse and longitudinal, for $R = 0$ only. Since we have seen that these two modes propel the sheet in opposite directions when acting separately, the magnitude and direction of propulsion when they act together must depend critically on the relative phase between the two modes.

Finally, we may calculate the rate of dissipation of energy in the fluid, as a generalization to finite R of Taylor's (1951) estimate at $R = 0$. If $\langle\dot{E}\rangle$ denotes the time-averaged rate of dissipation per unit area of the plate in the whole fluid, then to leading order in b

$$\langle\dot{E}\rangle = \mu\sigma^2 b^2 k[1 + F(R)] \tag{3.6}$$

for both transverse and longitudinal oscillations. This reduces to Taylor's result $2\mu\sigma^2b^2k$ at $R = 0$, and increases with R. In assessing efficiency of propulsion it is obviously more significant to consider U_∞ at fixed $\langle\dot{E}\rangle$ rather than fixed amplitude b, in which case we see that the propulsive velocity decreases rapidly with R for transverse oscillations. Even the longitudinal oscillation mode gives a slow decrease in efficiency as R increases, by this criterion.

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REFERENCES

Reynolds, A. J. 1965 *J. Fluid Mech.* **23**, 241.
Taylor, G. I. 1951 *Proc. Roy. Soc.* A **209**, 447.